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Double Hopf bifurcation and quasi-periodic attractors in delay-coupled limit cycle oscillators ☆

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ABSTRACT

We investigate the double Hopf bifurcation at zero equilibrium point. Firstly, we give the critical values of Hopf and double Hopf bifurcations. Secondly, we implement the normal form method and the center manifold theory for delay-coupled limit cycle oscillators, and derive the universal unfolding and a complete bifurcation diagram of the system. Thirdly, many interesting phenomena, such as attractive periodic motion and three-dimensional invariant torus, are observed using numerical simulation. Finally, the normal forms of several strong resonant cases are listed.

1. Introduction

The coupled system is a new field of science studying how parts of a system give rise to the collective behavior and how the system interacts with its environment. The coupled system is a method of studying complex systems and it concerns the topology between individuals. It can promote the awareness and understanding of coupled systems to discuss these complex behavior. Thus, a lot of researches have appeared in a wide range of subject areas, such as engineering [1], biology [2], social science [3], autonomous agents [4,5], neural networks [6] and so on. Coupled limit cycle oscillator models have been extensively studied in recent years because of the useful insight they provide into the collective behavior of many physical, chemical and biological systems. They may often produce many interesting new phenomena such as synchronization [7,8], phase trapping, phase locking, amplitude death, swarming [9], consensus, spatio-temporal chaos, etc. Amplitude death (also called oscillator death) refers to this quenching effect, where the oscillatory behavior of isolated units is suppressed and replaced by a stable equilibrium state in the coupled system. The first observation of the death phenomenon may be traced back to Lord Rayleigh's experiments on acoustics [10]. The differences in the intrinsic oscillation frequencies of the units can cause amplitude death [11]. In most of the physical and ecological systems, delays are common. The emergence of delay makes the dynamical systems become from finite dimension to infinite dimension, and induces more complex nonlinear dynamics. Therefore, delay-coupled nonlinear systems cause the concerns of many scholars. [12] has shown that delays can induce death even in a system with identical units. [13,14] obtained many valuable results about delays inducing death. Especially, [15] gave rigorous theoretical analysis about the conditions of amplitude death unlike the way of previous numerical analysis.

The following delay-coupled limit cycle oscillators have been investigated intensively.

$$\dot{Z}_1(t) = \left(1 + i\omega_1 - \left|Z_1(t)\right|^2\right)Z_1(t) + K\left[Z_2(t-\tau) - Z_1(t)\right],$$

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$$\dot{Z}_2(t) = \left(1 + i\omega_2 - \left|Z_2(t)\right|^2\right)Z_2(t) + K\left[Z_1(t-\tau) - Z_2(t)\right], \tag{1.1}$$

where $Z_j(t)$ is the complex amplitude of the jth oscillator with $j = 1, 2$. Each oscillator has a stable limit cycle of unit amplitude $|Z_j| = 1$ with angular frequency ω_j. $K \geqslant 0$ is the coupling strength and $\tau \geqslant 0$ is a measure of time delay. For a single oscillator

$$\dot{Z}(t) = \left(1 + i\omega - \left|Z(t)\right|^2\right)Z(t), \tag{1.2}$$

it has a stable limit cycle $z(t) = e^{i\omega t}$, which is so-called the Stuart–Landau equation, represents the normal form for a supercritical Hopf bifurcation. [16] has shown that there exist death islands for system (1.1), that is, zero equilibrium is from unstable to stable. And the boundaries of the islands can be found by considering the critical values of Hopf bifurcation (see [17,18]) and the methods by which [19,20] discussed the complex variable systems. Especially, we have given the amplitude death regions for system (1.1) in [18] and detected more complex dynamics at the intersection points of region boundaries by numerical methods. Here, we investigate the dynamics near the intersection points of region boundaries and hope to derive the perfect theoretical results. In fact, we need to consider a codimension-2 bifurcation, i.e. double Hopf bifurcation. Double Hopf bifurcation refers to the case that there are two different pairs of simply pure imaginary eigenvalues $\pm i\beta_\pm$ ($\beta_+ \geqslant \beta_- > 0$) at the same critical value. Noting $\beta_- : \beta_+ = k_1 : k_2$, the double Hopf bifurcation is called nonresonance if $k_1 : k_2$ is irrational [21,22], and it is called resonance if $k_1 : k_2$ is rational [23]. Furthermore, if $k_1 : k_2$ is rational, the bifurcation is a strong (low-order) resonance when $k_1 = 1$ and $k_2 = 1, 2, 3, 4$ [24,25], and a weak (high-order) resonance [26,27] otherwise. For double Hopf bifurcation, we usually can observe periodic and quasi-periodic motions, three-dimensional invariant torus, period doubling, homoclinic and heteroclinic connections, and tangles. Especially, strong resonant double Hopf bifurcation can present more complex nonlinear behavior because it may be a codimension-3 bifurcation.

Our paper is organized as follows: in Section 2, the existences of Hopf and double Hopf bifurcations are discussed for coupled system. In Section 3, the normal form method and the center manifold theory are used to analyze the double Hopf bifurcation for system. In Section 4, we simulate some interesting phenomena near the double Hopf bifurcation, such as periodic solutions and three-dimensional torus. In Section 5, the normal forms of several strong resonant cases are listed. In Section 6, we summarize our results.

2. Existences of Hopf and double Hopf bifurcations

We still follow the methods and results in [17,18]. Writing $Z_j(t) = x_j(t) + iy_j(t)$ $(j = 1, 2)$, Eq. (1.1) can be expressed in real form as

$$\begin{cases} \dot{x}_1(t) = (1-K)x_1(t) - \omega_1 y_1(t) + Kx_2(t-\tau) - x_1^3(t) - x_1(t)y_1^2(t), \\ \dot{y}_1(t) = \omega_1 x_1(t) + (1-K)y_1(t) + Ky_2(t-\tau) - x_1^2(t)y_1(t) - y_1^3(t), \\ \dot{x}_2(t) = (1-K)x_2(t) - \omega_2 y_2(t) + Kx_1(t-\tau) - x_2^3(t) - x_2(t)y_2^2(t), \\ \dot{y}_2(t) = \omega_2 x_2(t) + (1-K)y_2(t) + Ky_1(t-\tau) - x_2^2(t)y_2(t) - y_2^3(t). \end{cases} \tag{2.1}$$

Clearly, $(0, 0, 0, 0)$ is a fixed point. The characteristic equation of its corresponding linear system around the origin $(0, 0, 0, 0)$ is

$$\left[(\lambda - 1 + K)^2 - \omega_1\omega_2 - K^2 e^{-2\lambda\tau}\right]^2 + (\omega_1 + \omega_2)^2(\lambda - 1 + K)^2 = 0, \tag{2.2}$$

that is,

$$(\lambda - 1 + K)^2 - \omega_1\omega_2 - K^2 e^{-2\lambda\tau} = i(\omega_1 + \omega_2)(\lambda - 1 + K) \tag{2.3}$$

and

$$(\lambda - 1 + K)^2 - \omega_1\omega_2 - K^2 e^{-2\lambda\tau} = -i(\omega_1 + \omega_2)(\lambda - 1 + K). \tag{2.4}$$

It is not difficult to verify that $\alpha + i\beta$ is a root of (2.3) if and only if $\alpha - i\beta$ is a root of (2.4). So we only need to investigate Eq. (2.3).

When $\tau = 0$, the roots for Eq. (2.2) are

$$\lambda = 1 - K \pm \sqrt{K^2 - (\omega_1 - \omega_2)^2/4} \pm 2i\varpi.$$

When $\tau \neq 0$, let $\lambda = i\beta$ $(\beta \neq 0)$ be a root of Eq. (2.3), then β satisfies

$$\begin{cases} (1-K)^2 - (\beta - \omega_1)(\beta - \omega_2) = K^2\cos(2\beta\tau), \\ 2(1-K)(\beta - \frac{\omega_1+\omega_2}{2}) = K^2\sin(2\beta\tau). \end{cases} \tag{2.5}$$

From [17,18], we have the following theorem about the existence of Hopf bifurcation.

Theorem 2.1. *Suppose all the quadratic expressions make sense, system* (2.1) *undergoes a Hopf bifurcation at its zero equilibrium when* $\tau = \tau_n^j$ $(n = 1, 2, 3, 4,\ j = 0, 1, \ldots)$, *where*

$$\tau_1^j = \begin{cases} \frac{2j\pi + \arccos\frac{l_1}{K^2}}{2\beta_1}, & \text{if } l_1 \geqslant 0,\ K < 1, \\ \frac{2(j+1)\pi - \arccos\frac{l_1}{K^2}}{2\beta_1}, & \text{if } l_1 > 0,\ K > 1, \\ \frac{(2j+1)\pi - \arccos\frac{l_1}{K^2}}{2\beta_1}, & \text{if } l_1 < 0,\ K \leqslant 1, \\ \frac{(2j+1)\pi + \arccos\frac{l_1}{K^2}}{2\beta_1}, & \text{if } l_1 < 0,\ K > 1, \end{cases}$$

$$\tau_2^j = \begin{cases} \frac{2j\pi + \arccos\frac{l_2}{K^2}}{2\beta_2}, & \text{if } K < 1, \\ \frac{2(j+1)\pi - \arccos\frac{l_2}{K^2}}{2\beta_2}, & \text{if } K > 1, \end{cases}$$

$$\tau_3^j = \begin{cases} \frac{2(j+1)\pi - \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 > 0,\ K < 1, \\ \frac{2j\pi + \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 \geqslant 0,\ K > 1, \\ \frac{(2j+1)\pi + \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 < 0,\ K < 1, \\ \frac{(2j+1)\pi - \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 < 0,\ K \geqslant 1, \end{cases}$$

$$\tau_4^j = \begin{cases} \frac{2(j+1)\pi - \arccos\frac{l_2}{K^2}}{2\beta_4}, & \text{if } K < 1, \\ \frac{2j\pi + \arccos\frac{l_2}{K^2}}{2\beta_4}, & \text{if } K > 1, \end{cases}$$

when $\beta_{3,4} > 0$, *or*

$$\tau_3^j = \begin{cases} \frac{-2j\pi - \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 > 0,\ K < 1, \\ \frac{-2(j+1)\pi + \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 \geqslant 0,\ K > 1, \\ \frac{-(2j+1)\pi + \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 < 0,\ K < 1, \\ \frac{-(2j+1)\pi - \arccos\frac{l_1}{K^2}}{2\beta_3}, & \text{if } l_1 < 0,\ K \geqslant 1, \end{cases}$$

$$\tau_4^j = \begin{cases} \frac{-2j\pi - \arccos\frac{l_2}{K^2}}{2\beta_4}, & \text{if } K < 1, \\ \frac{-2(j+1)\pi + \arccos\frac{l_2}{K^2}}{2\beta_4}, & \text{if } K > 1, \end{cases}$$

when $\beta_{3,4} < 0$. *Here,*

$$\beta_1 = \frac{\omega_1 + \omega_2}{2} + \sqrt{\frac{(\omega_1 - \omega_2)^2}{4} - l_1 + (1 - K)^2},$$

$$\beta_2 = \frac{\omega_1 + \omega_2}{2} + \sqrt{\frac{(\omega_1 - \omega_2)^2}{4} - l_2 + (1 - K)^2},$$

$$\beta_3 = \frac{\omega_1 + \omega_2}{2} - \sqrt{\frac{(\omega_1 - \omega_2)^2}{4} - l_1 + (1 - K)^2},$$

$$\beta_4 = \frac{\omega_1 + \omega_2}{2} - \sqrt{\frac{(\omega_1 - \omega_2)^2}{4} - l_2 + (1 - K)^2},$$

where

$$l_1 = 2(1 - K)^2 - \sqrt{K^4 - (1 - K)^2(\omega_1 - \omega_2)^2},$$
$$l_2 = 2(1 - K)^2 + \sqrt{K^4 - (1 - K)^2(\omega_1 - \omega_2)^2}.$$

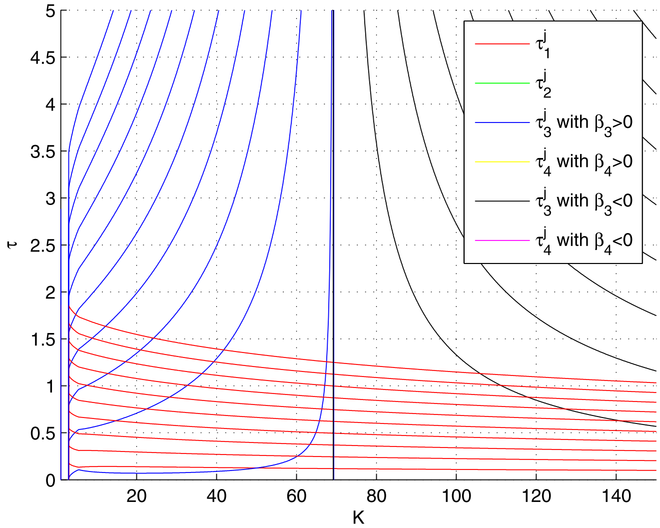

Fig. 1. The Hopf bifurcation curves for (1.1) at the trivial equilibrium when $\omega_1 = 10$ and $\omega_2 = 15$. The curves with different colors render τ_n^j with $n = 1, 2, 3, 4$, and j is from 0 to 9 from the bottom up.

The corresponding transverse condition is the following.

Remark 2.2.

$$\operatorname{sign} \frac{d\alpha}{d\tau}\bigg|_{\tau=\tau_n^j} = \begin{cases} 1, & \text{if } n = 1, \\ -1, & \text{if } n = 2, \\ -\operatorname{sign}\beta_3, & \text{if } n = 3, \\ \operatorname{sign}\beta_4, & \text{if } n = 4. \end{cases}$$

We can plot the critical values τ_n^j $(n = 1, 2, 3, 4;\ j = 0, 1, 2, \ldots)$ in the plane of K and τ in Fig. 1. We know that β_1 and β_3 exist when $\omega_1 = 10$ and $\omega_2 = 15$ from Fig. 1. There are some intersection points of τ_1^j and τ_3^j. In fact, double Hopf bifurcation occurs at these points.

3. Normal form for double Hopf bifurcation in coupled limit cycle oscillators without strong resonance

We have known that system (2.1) can undergo a double Hopf bifurcation. For making the dynamics of system (2.1) near the double Hopf bifurcation clear, in this section, we transform the infinite dimensional dynamic system into a finite dimensional system by the center manifold theory and simplify the system further by the normal form method.

From Fig. 1, if (2.2) has two pairs of roots with positive and negative real parts when $\tau = 0$, respectively, then its roots except $\pm i\beta_1$ and $\pm i\beta_3$ have negative real parts at the first intersection point by the transverse condition. Thus, the center manifold can be used to describe the dynamics of the whole system in this case.

Rescaling the time by $t \mapsto \frac{t}{\tau}$ to normalize the delay, Eq. (2.1) becomes

$$\begin{cases} \dot{x}_1(t) = \tau(1-K)x_1(t) - \tau\omega_1 y_1(t) + \tau K x_2(t-1) - \tau x_1^3(t) - \tau x_1(t) y_1^2(t), \\ \dot{y}_1(t) = \tau\omega_1 x_1(t) + \tau(1-K)y_1(t) + \tau K y_2(t-1) - \tau x_1^2(t) y_1(t) - \tau y_1^3(t), \\ \dot{x}_2(t) = \tau(1-K)x_2(t) - \tau\omega_2 y_2(t) + \tau K x_1(t-1) - \tau x_2^3(t) - \tau x_2(t) y_2^2(t), \\ \dot{y}_2(t) = \tau\omega_2 x_2(t) + \tau(1-K)y_2(t) + \tau K y_1(t-1) - \tau x_2^2(t) y_2(t) - \tau y_2^3(t). \end{cases} \tag{3.1}$$

Suppose that system (1.1) has eigenvalues $\pm i\beta_+$ and $\pm i\beta_-$ at (K_c, τ_c). Set $K = K_c + \mu_1$ and $\tau = \tau_c + \mu_2$ and choose the phase space $C = C([-1, 0], \mathbb{R}^4)$ as the Banach space of continuous functions from $[-1, 0]$ to $\mathbb{R}^4$ with the supremum norm. Following the notations in [28,29], for $\phi \in C$, we define

$$L_0\phi = \int_{-1}^{0} d\eta(\theta)\,\phi(\theta),$$

where

$$\eta(\theta)=\begin{cases}\tau_c A_1, & \theta=0,\\ 0, & \theta\in(-1,0),\\ -\tau_c B_1, & \theta=-1\end{cases}$$

with

$$A_1=\begin{pmatrix}1-K_c & -\omega_1 & 0 & 0\\ \omega_1 & 1-K_c & 0 & 0\\ 0 & 0 & 1-K_c & -\omega_2\\ 0 & 0 & \omega_2 & 1-K_c\end{pmatrix},\qquad B_1=K_c\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}.$$

Then the linearization equation at the trivial equilibrium of (3.1) is

$$\dot{X}(t)=L_0X_t,$$

and the bilinear form on $C^*\times C$ is

$$(\psi,\varphi)=\psi(0)\varphi(0)-\int_{-1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)\,d\eta(\theta)\,\varphi(\xi)\,d\xi,$$

where $\varphi(\theta)=(\varphi_1(\theta),\varphi_2(\theta),\varphi_3(\theta),\varphi_4(\theta))\in C$, $\psi(s)=(\psi_1(s),\psi_2(s),\psi_3(s),\psi_4(s))^T\in C^*$.

From the discussion given above, we know that the linearization of Eq. (3.1) has pure imaginary eigenvalues $\Lambda=\{\pm i\tau_c\beta_+,\pm i\tau_c\beta_-\}$ at the double Hopf bifurcation point and the other eigenvalues with negative real parts. We then designate $P_\Lambda\subset C$ as the 4-dimensional center subspace spanned by the basis vectors of the linear operator L_0 associated with the imaginary characteristic roots. We decompose C as $C=P_\Lambda\oplus Q_\Lambda$, where Q_Λ is the complement subspace of P_Λ. Referring to [28–30], Eq. (3.1) can be rewritten as an ordinary differential equation in the Banach space BC of functions bounded and continuous on $[-1,0)$ with a possible jump discontinuity at 0. Elements of BC are of the form $\phi+X_0\alpha$, where $\phi\in C$, $\alpha\in\mathbb{R}^4$ and $X_0(\theta)=0$ for $\theta\in[-1,0)$ and $X_0(0)=I$. Let $\pi:BC\to P_\Lambda$ be a continuous projection defined by $\pi(\phi+X_0\alpha)=\Phi[(\Psi,\phi)+\Psi(0)\alpha]$.

We suppose that the bases of P_Λ and P^*_Λ, respectively, are

$$\Phi(\theta)=\big(q_1(\theta),\bar{q}_1(\theta),q_2(\theta),\bar{q}_2(\theta)\big)$$

and

$$\Psi(s)=\begin{pmatrix}q_1^*(s)\\ \bar{q}_1^*(s)\\ q_2^*(s)\\ \bar{q}_2^*(s)\end{pmatrix}\doteq(\psi_{ij})_{i,j=1,2,3,4}.$$

It can be computed directly that

$$\begin{aligned}
q_1(\theta)&=(K_c,-iK_c,c,-ic)^Te^{i\tau_c\beta_+\theta},\\
q_2(\theta)&=(K_c,-iK_c,d,-id)^Te^{i\tau_c\beta_-\theta},\\
q_1^*(s)&=\frac{1}{m_1n_2-m_2n_1}\Big(K_c\big(n_2e^{-i\tau_c\beta_+s}-m_2e^{-i\tau_c\beta_-s}\big),iK_c\big(n_2e^{-i\tau_c\beta_+s}-m_2e^{-i\tau_c\beta_-s}\big),\\
&\quad n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+(1-s)}-m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-(1-s)},\\
&\quad i\big[n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+(1-s)}-m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-(1-s)}\big]\Big),\\
q_2^*(s)&=\frac{1}{m_1n_2-m_2n_1}\Big(K_c\big(m_1e^{-i\tau_c\beta_-s}-n_1e^{-i\tau_c\beta_+s}\big),iK_c\big(m_1e^{-i\tau_c\beta_-s}-n_1e^{-i\tau_c\beta_+s}\big),\\
&\quad m_1\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-(1-s)}-n_1\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+(1-s)},\\
&\quad i\big[m_1\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-(1-s)}-n_1\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+(1-s)}\big]\Big),
\end{aligned}$$

where

$$\begin{aligned}
c&=\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+},\\
d&=\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-},\\
m_1&=2\big[K_c^2+\big(K_c-1+i(\beta_+-\omega_1)\big)^2e^{2i\tau_c\beta_+}+K_c^2\tau_c\big(K_c-1+i(\beta_+-\omega_1)\big)\big],\\
m_2&=2\big[K_c^2+\big(K_c-1+i(\beta_+-\omega_1)\big)\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c(\beta_++\beta_-)}\big]-2K_c^2\tau_ce^{-i\tau_c\beta_+}
\end{aligned}$$

$$\times \big[\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}+\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big]\int_0^{-1} e^{i\tau_c(\beta_--\beta_+)\xi}\,d\xi,$$

$$n_1=2\big[K_c^2+\big(K_c-1+i(\beta_+-\omega_1)\big)\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c(\beta_++\beta_-)}\big]-2K_c^2\tau_c e^{-i\tau_c\beta_-}$$

$$\times \big[\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}+\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big]\int_0^{-1} e^{i\tau_c(\beta_+-\beta_-)\xi}\,d\xi,$$

$$n_2=2\big[K_c^2+\big(K_c-1+i(\beta_--\omega_1)\big)^2e^{2i\tau_c\beta_-}+K_c^2\tau_c\big(K_c-1+i(\beta_--\omega_1)\big)\big].$$

Let $\mu=(\mu_1,\mu_2)$ and Eq. (3.1) can be written as

$$\dot X(t)=L(\mu)X_t+F(X_t,\mu), \tag{3.2}$$

where

$$L(\mu)X_t=(\tau_c+\mu_2)\begin{pmatrix}1-(K_c+\mu_1) & -\omega_1 & 0 & 0\\ \omega_1 & 1-(K_c+\mu_1) & 0 & 0\\ 0 & 0 & 1-(K_c+\mu_1) & -\omega_2\\ 0 & 0 & \omega_2 & 1-(K_c+\mu_1)\end{pmatrix}\begin{pmatrix}x_{1t}(0)\\ y_{1t}(0)\\ x_{2t}(0)\\ y_{2t}(0)\end{pmatrix}$$
$$+(K_c+\mu_1)(\tau_c+\mu_2)\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}\begin{pmatrix}x_{1t}(-1)\\ y_{1t}(-1)\\ x_{2t}(-1)\\ y_{2t}(-1)\end{pmatrix},$$

and

$$F(X_t,\mu)=-(\tau_c+\mu_2)\begin{pmatrix}x_{1t}^3(0)+x_{1t}(0)y_{1t}^2(0)\\ x_{1t}^2(0)y_{1t}(0)+y_{1t}^3(0)\\ x_{2t}^3(0)+x_{2t}(0)y_{2t}^2(0)\\ x_{2t}^2(0)y_{2t}(0)+y_{2t}^3(0)\end{pmatrix}.$$

In BC, Eq. (3.2) becomes an abstract ODE,

$$\frac{d}{dt}u=Au+X_0\tilde F, \tag{3.3}$$

where $u\in C$, and A is defined by

$$A: C^1\to BC,\qquad Au=\dot u+X_0\big[L_0u-\dot u(0)\big],$$

and

$$\tilde F(u,\mu)=\big[L(\mu)-L_0\big]u+F(u,\mu).$$

By the continuous projection π, we can decompose the enlarged phase space by $\Lambda=\{\pm i\tau_c\beta_+,\pm i\tau_c\beta_-\}$ as $BC=P\oplus \operatorname{Ker}\pi$. Let $u_t=\Phi z(t)+v(t)$, Eq. (3.3) is therefore decomposed as the system

$$\begin{aligned}&\dot z=Bz+\Psi(0)\tilde F(\Phi z+v,\mu),\\ &\dot v=A_{Q^1}v+(I-\pi)X_0\tilde F(\Phi z+v,\mu),\end{aligned} \tag{3.4}$$

where $y\in Q^1:=Q\cap C^1\subset \operatorname{Ker}\pi$. A_{Q^1} is the restriction of A as an operator from Q^1 to the Banach space $\operatorname{Ker}\pi$. Neglecting higher order terms with respect to parameters μ_1 and μ_2, Eq. (3.4) can be written as

$$\begin{aligned}&\dot z_1=i\tau_c\beta_+z_1+\Sigma_{j=1}^4\psi_{1j}\big(F_2^j+F_3^j\big)+h.o.t.,\\ &\dot z_2=-i\tau_c\beta_+z_2+\Sigma_{j=1}^4\psi_{2j}\big(F_2^j+F_3^j\big)+h.o.t.,\\ &\dot z_3=i\tau_c\beta_-z_3+\Sigma_{j=1}^4\psi_{3j}\big(F_2^j+F_3^j\big)+h.o.t.,\\ &\dot z_4=-i\tau_c\beta_-z_4+\Sigma_{j=1}^4\psi_{4j}\big(F_2^j+F_3^j\big)+h.o.t.,\\ &\dot v=A_{Q^1}v+(I-\pi)X_0\tilde F(\Phi z+v,\mu),\end{aligned}$$

where

$$\begin{aligned}
F_2^1 &= \big(\mu_2(1-K_c)-\mu_1\tau_c\big)\big(K_c z_1+K_c z_2+K_c z_3+K_c z_4+v_1(0)\big)\\
&\quad -\omega_1\mu_2\big(-iK_c z_1+iK_c z_2-iK_c z_3+iK_c z_4+v_2(0)\big)\\
&\quad +(\tau_c\mu_1+K_c\mu_2)\big(ce^{-i\tau_c\beta_+}z_1+\bar{c}e^{i\tau_c\beta_+}z_2+de^{-i\tau_c\beta_-}z_3+\bar{d}e^{i\tau_c\beta_-}z_4+v_3(-1)\big),\\
F_2^2 &= \big(\mu_2(1-K_c)-\mu_1\tau_c\big)\big(-iK_c z_1+iK_c z_2-iK_c z_3+iK_c z_4+v_2(0)\big)\\
&\quad +\omega_1\mu_2\big(K_c z_1+K_c z_2+K_c z_3+K_c z_4+v_1(0)\big)\\
&\quad +(\tau_c\mu_1+K_c\mu_2)\big(-ice^{-i\tau_c\beta_+}z_1+i\bar{c}e^{i\tau_c\beta_+}z_2-ide^{-i\tau_c\beta_-}z_3+i\bar{d}e^{i\tau_c\beta_-}z_4+v_4(-1)\big),\\
F_2^3 &= \big(\mu_2(1-K_c)-\mu_1\tau_c\big)\big(cz_1+\bar{c}z_2+dz_3+\bar{d}z_4+v_3(0)\big)\\
&\quad -\omega_2\mu_2\big(-icz_1+i\bar{c}z_2-idz_3+i\bar{d}z_4+v_4(0)\big)\\
&\quad +(\tau_c\mu_1+K_c\mu_2)\big(K_ce^{-i\tau_c\beta_+}z_1+K_ce^{i\tau_c\beta_+}z_2+K_ce^{-i\tau_c\beta_-}z_3+K_ce^{i\tau_c\beta_-}z_4+v_1(-1)\big),\\
F_2^4 &= \big(\mu_2(1-K_c)-\mu_1\tau_c\big)\big(-icz_1+i\bar{c}z_2-idz_3+i\bar{d}z_4+v_4(0)\big)\\
&\quad +\omega_2\mu_2\big(cz_1+\bar{c}z_2+dz_3+\bar{d}z_4+v_3(0)\big)\\
&\quad +(\tau_c\mu_1+K_c\mu_2)\big(-iK_ce^{-i\tau_c\beta_+}z_1+iK_ce^{i\tau_c\beta_+}z_2-iK_ce^{-i\tau_c\beta_-}z_3+iK_ce^{i\tau_c\beta_-}z_4+v_2(-1)\big),\\
F_3^1 &= -\tau_c\big[\big(K_c z_1+K_c z_2+K_c z_3+K_c z_4+v_1(0)\big)^3\\
&\quad +\big(K_c z_1+K_c z_2+K_c z_3+K_c z_4+v_1(0)\big)\big(-iK_c z_1+iK_c z_2+-iK_c z_3+iK_c z_4+v_2(0)\big)^2\big],\\
F_3^2 &= -\tau_c\big[\big(K_c z_1+K_c z_2+K_c z_3+K_c z_4+v_1(0)\big)^2\big(-iK_c z_1+iK_c z_2-iK_c z_3+iK_c z_4+v_2(0)\big)\\
&\quad +\big(-iK_c z_1+iK_c z_2-iK_c z_3+iK_c z_4+v_2(0)\big)^3\big],\\
F_3^3 &= -\tau_c\big[\big(cz_1+\bar{c}z_2+dz_3+\bar{d}z_4+v_3(0)\big)^3\\
&\quad +\big(cz_1+\bar{c}z_2+dz_3+\bar{d}z_4+v_3(0)\big)\big(-icz_1+i\bar{c}z_2-idz_3+i\bar{d}z_4+v_4(0)\big)^2\big],\\
F_3^4 &= -\tau_c\big[\big(cz_1+\bar{c}z_2+dz_3+\bar{d}z_4+v_3(0)\big)^2\big(-icz_1+i\bar{c}z_2-idz_3+i\bar{d}z_4+v_4(0)\big)\\
&\quad +\big(-icz_1+i\bar{c}z_2-idz_3+i\bar{d}z_4+v_4(0)\big)^3\big].
\end{aligned}$$

Let M_2 denote the operator defined in $V_2^6(\mathbb{C}^4\times \operatorname{Ker}\pi)$, with

$$M_2^1 : V_2^6\big(\mathbb{C}^4\big)\mapsto V_2^6\big(\mathbb{C}^4\big),$$

and

$$\big(M_2^1 p\big)(z,\mu)=D_z p(z,\mu)Bz-Bp(z,\mu),$$

where $V_2^6(\mathbb{C}^4)$ denotes the linear space of the second order homogeneous polynomials in six variables $(z_1,z_2,z_3,z_4,\mu_1,\mu_2)$, and with coefficients in $\mathbb{C}^4$. If we do not consider the strong resonant cases, it is easy to check that one may choose the decomposition

$$V_2^6\big(\mathbb{C}^4\big)=\operatorname{Im}\big(M_1^2\big)\oplus\operatorname{Im}\big(M_1^2\big)^c$$

with complementary space

$$\operatorname{Im}\big(M_2^1\big)^c=\operatorname{span}\left\{\begin{pmatrix} z_1\mu_i\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\z_2\mu_i\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\z_3\mu_i\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\z_4\mu_i\end{pmatrix}\right\},\quad i=1,2.$$

Then the normal form of Eq. (3.2) on the center manifold of the origin near $\mu=0$ has the form

$$\dot{z}=Bz+\frac{1}{2}g_2^1(z,0,\mu)+h.o.t.,\tag{3.5}$$

where g_2^1 is the function giving the quadratic terms in (z,μ) for $v=0$, and is determined by $g_2^1(z,0,\mu)=\operatorname{Proj}_{\operatorname{Im}(M_2^1)^c}\times f_2^1(z,0,\mu)$, where $f_2^1(z,0,\mu)$ is the function giving the quadratic terms in (z,μ) for $v=0$ defined by the first equation of (3.4). Then, the normal form in Eq. (2.1) is truncated to the second order, as

$$\begin{aligned}
\dot{z}_1 &= i\tau_c\beta_+ z_1 + \lambda_1 z_1 + h.o.t.,\\
\dot{z}_2 &= -i\tau_c\beta_+ z_2 + \bar{\lambda}_1 z_2 + h.o.t.,\\
\dot{z}_3 &= i\tau_c\beta_- z_3 + \lambda_2 z_3 + h.o.t.,\\
\dot{z}_4 &= -i\tau_c\beta_- z_4 + \bar{\lambda}_2 z_4 + h.o.t.,
\end{aligned}\tag{3.6}$$

where

$$\begin{aligned}
\lambda_1 &= \frac{2K_c(n_2-m_2)}{m_1n_2-m_2n_1}\big[\big(ce^{-i\tau_c\beta_+}-K_c\big)\tau_c\mu_1+\big(1-K_c+i\omega_1+ce^{-i\tau_c\beta_+}\big)K_c\mu_2\big]\\
&\quad+\frac{2[n_2(K_c-1+i(\beta_+-\omega_1))e^{i\tau_c\beta_+}-m_2(K_c-1+i(\beta_--\omega_1))e^{i\tau_c\beta_-}]}{m_1n_2-m_2n_1}\\
&\quad\times\big[\big(K_ce^{-i\tau_c\beta_+}-c\big)\tau_c\mu_1+\big((1-K_c)c+i\omega_2c+K_c^2e^{-i\tau_c\beta_+}\big)\mu_2\big],\\
\lambda_2 &= \frac{2K_c(m_1-n_1)}{m_1n_2-m_2n_1}\big[\big(de^{-i\tau_c\beta_-}-K_c\big)\tau_c\mu_1+\big(1-K_c+i\omega_1+de^{-i\tau_c\beta_-}\big)K_c\mu_2\big]\\
&\quad+\frac{2[m_1(K_c-1+i(\beta_--\omega_1))e^{i\tau_c\beta_-}-n_1(K_c-1+i(\beta_+-\omega_1))e^{i\tau_c\beta_+}]}{m_1n_2-m_2n_1}\\
&\quad\times\big[\big(K_ce^{-i\tau_c\beta_-}-d\big)\tau_c\mu_1+\big((1-K_c)d+i\omega_2d+K_c^2e^{-i\tau_c\beta_-}\big)\mu_2\big].
\end{aligned}$$

To find the third-order normal form, let M_3 denote the operator defined in $V_3^6(\mathbb{C}^4\times\operatorname{Ker}\pi)$, with

$$M_3^1: V_3^6\big(\mathbb{C}^4\big)\mapsto V_3^6\big(\mathbb{C}^4\big),$$

and

$$\big(M_3^1p\big)(z,\mu)=D_zp(z,\mu)Bz-Bp(z,\mu),$$

where $V_3^6(\mathbb{C}^4)$ denotes the linear space of the third order homogeneous polynomials in four variables (z_1,z_2,z_3,z_4), and with coefficients in $\mathbb{C}^4$. Similarly, when there is no strong resonance, it is easy to check that one may choose the decomposition

$$V_3^6\big(\mathbb{C}^4\big)=\operatorname{Im}\big(M_3^1\big)\oplus\operatorname{Im}\big(M_3^1\big)^c$$

with complementary space

$$\operatorname{Im}\big(M_3^1\big)^c=\operatorname{span}\left\{\begin{pmatrix}z_1^2z_2\\0\\0\\0\end{pmatrix},\begin{pmatrix}z_1z_3z_4\\0\\0\\0\end{pmatrix},\begin{pmatrix}0\\z_1z_2^2\\0\\0\end{pmatrix},\begin{pmatrix}0\\z_2z_3z_4\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\z_1z_2z_3\\0\end{pmatrix},\begin{pmatrix}0\\0\\z_3^2z_4\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\z_1z_2z_4\end{pmatrix},\begin{pmatrix}0\\0\\0\\z_3z_4^2\end{pmatrix}\right\}.$$

Then we can derive the normal form up to the third order

$$\dot{z}=Bz+\frac{1}{2}g_2^1(z,0,\mu)+\frac{1}{3!}g_3^1(z,0,0)+h.o.t.,\tag{3.7}$$

where

$$\frac{1}{3!}g_3^1(z,0,0)=\frac{1}{3!}\big(I-P_{I,3}^1\big)f_3^1(z,0,0),$$

and $f_3^1(z,0,0)$ is the function giving the cubic terms in (z,μ,v) for $\mu=0, v=0$ defined by the first equation of (3.4). Then, Eq. (3.7) can be written as

$$\begin{aligned}
\dot{z}_1 &= i\tau_c\beta_+ z_1+\lambda_1z_1+a_{11}z_1^2z_2+a_{12}z_1z_3z_4+h.o.t.,\\
\dot{z}_2 &= -i\tau_c\beta_+ z_2+\bar{\lambda}_1z_2+\bar{a}_{11}z_1z_2^2+\bar{a}_{12}z_2z_3z_4+h.o.t.,\\
\dot{z}_3 &= i\tau_c\beta_- z_3+\lambda_2z_3+a_{21}z_3^2z_4+a_{22}z_1z_2z_3+h.o.t.,\\
\dot{z}_4 &= -i\tau_c\beta_- z_4+\bar{\lambda}_2z_4+\bar{a}_{21}z_3z_4^2+\bar{a}_{22}z_1z_2z_4+h.o.t.,
\end{aligned}\tag{3.8}$$

where

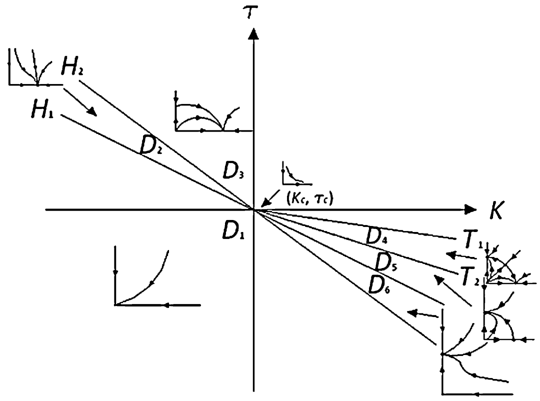

Fig. 2. The bifurcation diagram and phase portraits for (3.9) with parameter (K,τ) near the coordinate origin $K_c=50$, $\tau_c=0.1257$ when $\omega_1=10$, $\omega_2=15$.

$$a_{11}=-\frac{8\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(n_2-m_2)+c|c|^2\big(n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}-m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big)\big],$$
$$a_{12}=-\frac{16\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(n_2-m_2)+c|d|^2\big(n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}-m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big)\big],$$
$$a_{21}=-\frac{8\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(m_1-n_1)+d|d|^2\big(m_1\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}-n_1\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}\big)\big],$$
$$a_{22}=-\frac{16\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(m_1-n_1)+d|c|^2\big(m_1\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}-n_1\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}\big)\big].$$

In polar coordinates $z_1=r_1e^{-i\rho_1}$, $z_2=r_1e^{i\rho_1}$, $z_3=r_2e^{-i\rho_2}$ and $z_4=r_2e^{i\rho_2}$, the amplitude equation resulted from Eq. (3.8) is

$$\begin{cases} r_1'=(\alpha_1+\bar{a}r_1^2+\bar{b}r_2^2)r_1, \\ r_2'=(\alpha_2+\bar{c}r_1^2+\bar{d}r_2^2)r_2, \end{cases} \tag{3.9}$$

where

$$\alpha_1=\operatorname{Re}\lambda_1,\qquad \alpha_2=\operatorname{Re}\lambda_2,\qquad \bar{a}=\operatorname{Re}a_{11},\qquad \bar{b}=\operatorname{Re}a_{12},\qquad \bar{c}=\operatorname{Re}a_{21},\qquad \bar{d}=\operatorname{Re}a_{22}.$$

Notating $\theta=\frac{\bar{b}}{\bar{d}}$ and $\delta=\frac{\bar{c}}{\bar{a}}$, the parametric portraits of (3.9) on the plane of α_1 and α_2 can be referred in [31,32] ("simple" and "difficult" cases).

Referring to [29,31,33] and similarly to [34,35], we can give some analyses and examples to observe the behavior near the double Hopf bifurcation. Choosing $\omega_1=10$, $\omega_2=15$, we can obtain $K_c=50$, $\tau_c=0.1257$ from Fig. 1 and compute that $\beta_+=22.1548$, $\beta_-=2.8452$. The unfolding parameters are $\alpha_1=0.0058\mu_1+290.4624\mu_2$ and $\alpha_2=0.0011\mu_1+7.7329\mu_2$. The coefficients of Eq. (3.9) are $\bar{a}=-2.4224\times10^{11}$, $\bar{b}=-6.1726\times10^{11}$, $\bar{c}=-4.3717\times10^{11}$, $\bar{d}=-5.8303\times10^{11}$, so $\theta=1.0587$, $\delta=1.8047$ and $\theta\delta=1.9106$.

Thus, the case I ("simple" case) happens under the parameters above and the corresponding bifurcation diagram and phase portraits are given by Fig. 2.

In Fig. 2, the parameter plane (K,τ) can be divided into six regions. In region D_1, the trivial equilibrium is a sink; when the parameters vary across the line H_1 from D_1 to D_2, the trivial equilibrium becomes a saddle point and an asymptotically stable limit cycle (sink) is bifurcated; in region D_3, the trivial equilibrium becomes a source and another saddle limit cycle is bifurcated; in region D_4, a unstable quasi-periodic solution is bifurcated and the saddle limit cycle becomes a sink; in region D_5, the quasi-periodic solution above disappears and a sink limit cycle becomes a saddle; in region D_6, the saddle limit cycle disappears and the trivial equilibrium becomes a saddle point. In these phase portraits of Fig. 2, the horizontal and the vertical axes are r_1 and r_2 coordinates, respectively. Here, in Fig. 2, the bifurcation critical lines are, respectively,

$$H_1:\quad \tau=0.1257-1.9968\times10^{-5}(K-50),$$
$$H_2:\quad \tau=0.1257-1.4225\times10^{-4}(K-50),$$
$$T_1:\quad \tau=0.1257-1.6422\times10^{-5}(K-50)+O\big(\big(0.0011(K-50)+7.7329(\tau-0.1257)\big)^2\big)$$

with $\tau>0.1257-1.4225\times10^{-4}(K-50)$,

$$T_2:\quad \tau=0.1257-1.8137\times10^{-5}(K-50)+O\big(\big(0.0058(K-50)+290.4626(\tau-0.1257)\big)^2\big)$$

with $\tau>0.1257-1.9968\times10^{-5}(K-50)$.

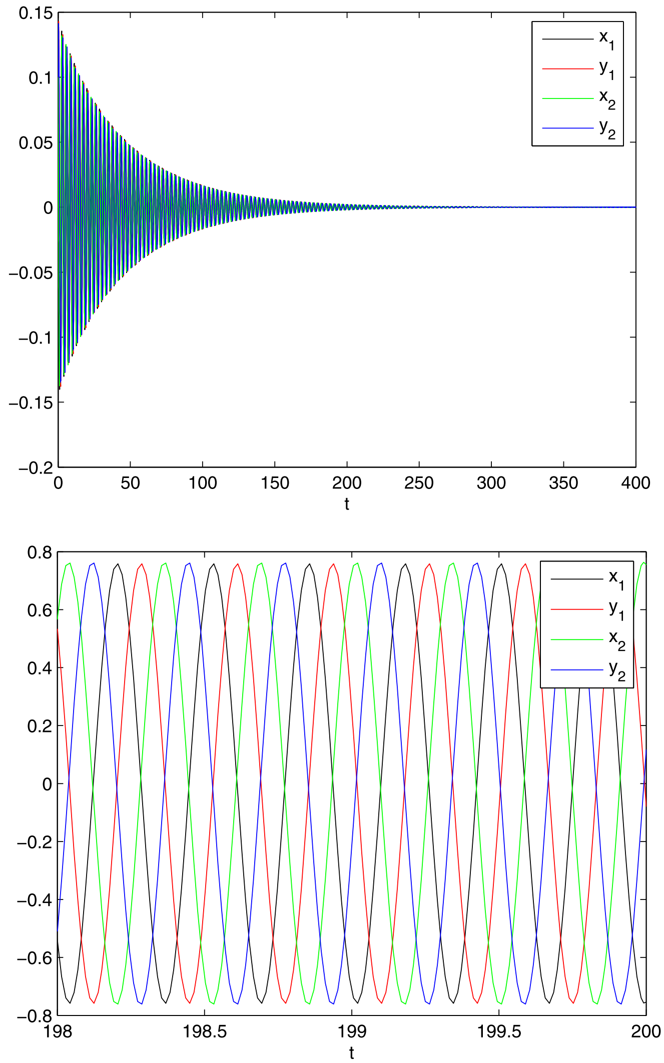

Fig. 3. Asymptotically stable trivial equilibrium and periodic solution of system (1.1) with $(K,\tau)=(49.97, 0.0957)\in D_1$ and $(50.03, 0.1557)\in D_3$, respectively.

4. Periodic solution and three-dimensional tori

The above analytical information shows that the trivial equilibrium is asymptotically stable in region D_1. Taking $K=50+\mu_1$, $\tau=0.1257+\mu_2$, where $\mu_1=-0.03$, $\mu_2=-0.03$, then $(K,\tau)\in D_1$. The upper figure of Fig. 3 shows that Eq. (1.1) has an attractive trivial equilibrium. Choosing $\mu_1=0.03, \mu_2=0.03$, $(K,\tau)\in D_3$ in which there exists an asymptotically stable periodic solution (see the lower figure of Fig. 3).

In addition, we simulate a three-dimensional invariant torus (see Fig. 4, where $\omega_1=1$, $\omega_2=3$, $K=0.6$ and $\tau=0.1$) although we do not obtain the corresponding parameters which make the center manifold stable. By choosing the Poincaré section given by $x_1(t)=0$, the closed curve on the Poincaré section verifies the existence of the corresponding asymptotically stable three-dimensional torus.

5. Normal forms of several strong resonant cases

Through the implementation process of the center manifold and the normal form above, we find that the corresponding complementary space is different from Section 3 which induces the normal form is more complex than (3.8) when the following strong resonant cases occur.

(i) When $\beta_-:\beta_+=1:3$, the corresponding complementary space is (we do not repeat the same space as the previous one)

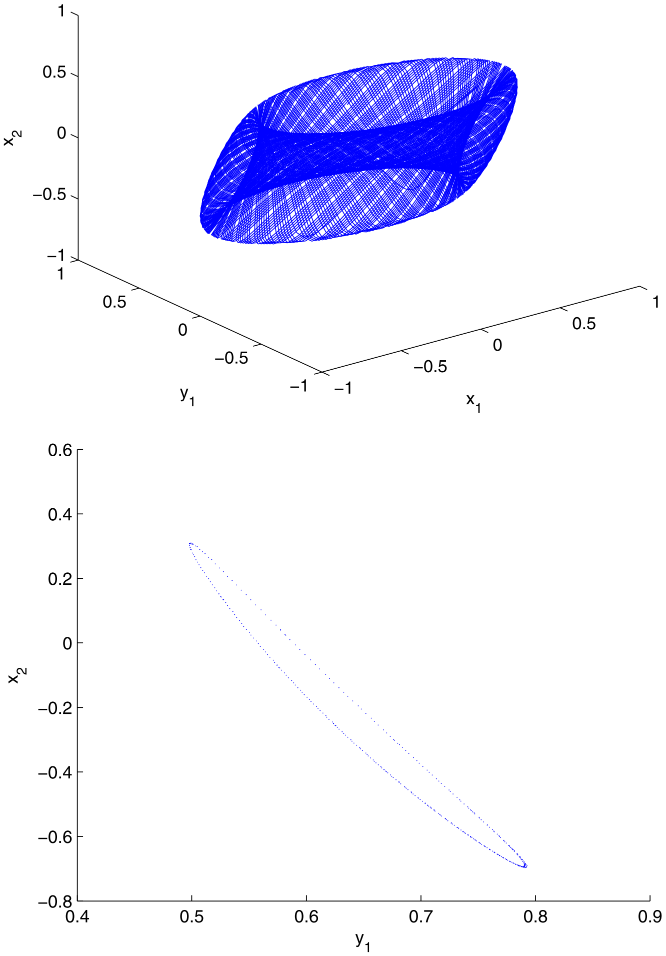

Fig. 4. The three-dimensional torus and Poincaré section of system (2.1) with $\omega_1 = 1$, $\omega_2 = 3$, $K = 0.6$ and $\tau = 0.1$.

$$\operatorname{Im}\left(M_3^1\right)^c = \operatorname{span}\left\{ \begin{pmatrix} z_1^2 z_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1 z_3 z_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1 z_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2 z_3 z_4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1 z_2 z_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_3^2 z_4 \\ 0 \end{pmatrix}, \right.$$
$$\left. \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1 z_2 z_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_3 z_4^2 \end{pmatrix}, \begin{pmatrix} z_3^3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_4^3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1 z_4^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_2 z_3^2 \end{pmatrix} \right\}.$$

Thus, the normal form corresponding to (3.8) is

$$\begin{aligned}
\dot z_1 &= i\tau_c\beta_+ z_1 + \lambda_1 z_1 + a_{11} z_1^2 z_2 + a_{12} z_1 z_3 x_4 + h.o.t.,\\
\dot z_2 &= -i\tau_c\beta_+ z_2 + \bar\lambda_1 z_2 + \bar a_{11} z_1 z_2^2 + \bar a_{12} z_2 z_3 z_4 + h.o.t.,\\
\dot z_3 &= i\tau_c\beta_- z_3 + \lambda_2 z_3 + a_{21} z_3^2 z_4 + a_{22} z_1 z_2 z_3 + a_{23} z_1 z_4^2 + h.o.t.,\\
\dot z_4 &= -i\tau_c\beta_- z_4 + \bar\lambda_2 z_4 + \bar a_{21} z_3 z_4^2 + \bar a_{22} z_1 z_2 z_4 + \bar a_{23} z_2 z_3^2 + h.o.t.,
\end{aligned} \tag{5.1}$$

where

$$a_{23} = -4\tau_c \frac{K_c^3(m_1 - n_1) + c\bar d^2[m_1(K_c - 1 - i(\beta_- - \omega_1))e^{i\tau_c\beta_-} - n_1(K_c - 1 - i(\beta_+ - \omega_1))e^{i\tau_c\beta_+}]}{m_1 n_2 - m_2 n_1}.$$

(ii) When $\beta_- : \beta_+ = 1 : 1$, the corresponding complementary spaces are

$$\mathrm{Im}(M_2^1)^c = \mathrm{span}\left\{ \begin{pmatrix} z_1\mu_i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2\mu_i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_3\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_4\mu_i \end{pmatrix}, \begin{pmatrix} z_3\mu_i \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_4\mu_i \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1\mu_i \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_2\mu_i \end{pmatrix} \right\},$$
$$i = 1, 2,$$

and

$$\begin{aligned} \mathrm{Im}(M_3^1)^c = \mathrm{span}\Bigg\{ & \begin{pmatrix} z_1^2 z_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1z_3z_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1z_2^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_2z_3z_4 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1z_2z_3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_3^2z_4 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1z_2z_4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_3z_4^2 \end{pmatrix}, \begin{pmatrix} z_1^2z_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z_1z_2z_3 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z_2z_3^2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} z_3^2z_4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1z_4^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_1z_2z_4 \\ 0 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ z_2^2z_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ z_3z_4^2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1^2z_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1z_3z_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_1^2z_4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ z_2z_3^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1z_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_2z_3z_4 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_1z_4^2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ z_2^2z_3 \end{pmatrix} \Bigg\}. \end{aligned}$$

Thus, the normal form corresponding to (3.8) is

$$\begin{aligned} \dot{z}_1 &= i\tau_c\beta_+ z_1 + \lambda_1 z_1 + \nu_1 z_3 + a_{11}z_1^2z_2 + a_{12}z_1z_3z_4 + a_{14}z_1^2z_4 + a_{15}z_3^2z_4 + a_{16}z_2z_3^2 + a_{17}z_1z_2z_3 + h.o.t., \\ \dot{z}_2 &= -i\tau_c\beta_+ z_2 + \bar{\lambda}_1 z_2 + \bar{\nu}_1 z_4 + \bar{a}_{11}z_1z_2^2 + \bar{a}_{12}z_2z_3z_4 + \bar{a}_{14}z_2^2z_3 + \bar{a}_{15}z_3z_4^2 + \bar{a}_{16}z_1z_4^2 + \bar{a}_{17}z_1z_2x_4 + h.o.t., \\ \dot{z}_3 &= i\tau_c\beta_- z_3 + \lambda_2 z_3 + \nu_2 z_1 + a_{21}z_3^2z_4 + a_{22}z_1z_2z_3 + a_{24}z_1^2z_2 + a_{25}z_1^2z_4 + a_{26}z_2z_3^2 + a_{27}z_1z_3z_4 + h.o.t., \\ \dot{z}_4 &= -i\tau_c\beta_- z_4 + \bar{\lambda}_2 z_4 + \bar{\nu}_2 z_2 + \bar{a}_{21}z_3z_4^2 + \bar{a}_{22}z_1z_2z_4 + \bar{a}_{24}z_1z_2^2 + \bar{a}_{25}z_2^2z_3 + \bar{a}_{26}z_1z_4^2 + \bar{a}_{27}z_2z_3z_4 + h.o.t., \end{aligned} \tag{5.2}$$

where

$$\begin{aligned} \nu_1 &= \frac{2K_c(n_2-m_2)}{m_1n_2-m_2n_1}\big[(d-K_c)\tau_c\mu_1 + (1-K_c+i\omega_1+d)K_c\mu_2\big]\big[(d-K_c)\tau_c\mu_1 + \big((1-K_c)d \\ &\quad + i\omega_2 d + K_c^2\big)\mu_2\big] + \frac{2[n_2(K_c-1+i(\beta_+-\omega_1))e^{i\tau_c\beta_+} - m_2(K_c-1+i(\beta_--\omega_1))e^{i\tau_c\beta_-}]}{m_1n_2-m_2n_1}, \\ \nu_2 &= \frac{2K_c(m_1-n_1)}{m_1n_2-m_2n_1}\big[(c-K_c)\tau_c\mu_1 + (1-K_c+i\omega_1+c)K_c\mu_2\big]\big[(c-K_c)\tau_c\mu_1 + \big((1-K_c)c \\ &\quad + i\omega_2 c + K_c^2\big)\mu_2\big] + \frac{2[m_1(K_c-1+i(\beta_--\omega_1))e^{i\tau_c\beta_-} - n_1(K_c-1+i(\beta_+-\omega_1))e^{i\tau_c\beta_+}]}{m_1n_2-m_2n_1}, \\ a_{14} &= -\frac{8\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(n_2-m_2) + c^2\bar{d}\big(n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+} - m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big)\big], \\ a_{15} &= -\frac{8\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(n_2-m_2) + d|d|^2\big(n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+} - m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big)\big], \\ a_{16} &= -\frac{8\tau_c}{m_1n_2-m_2n_1}\big[c\bar{d}^2\big(n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+} - m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big)\big], \\ a_{17} &= -\frac{16\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(n_2-m_2) + |c|^2 d\big(n_2\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+} - m_2\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-}\big)\big], \\ a_{24} &= -\frac{8\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(m_1-n_1) + c|c|^2\big(m_1\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-} - n_1\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}\big)\big], \\ a_{25} &= -\frac{8\tau_c}{m_1n_2-m_2n_1}\big[K_c^4(m_1-n_1) + c^2\bar{d}\big(m_1\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-} - n_1\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}\big)\big], \\ a_{26} &= -\frac{8\tau_c}{m_1n_2-m_2n_1}\big[c\bar{d}^2\big(m_1\big(K_c-1+i(\beta_--\omega_1)\big)e^{i\tau_c\beta_-} - n_1\big(K_c-1+i(\beta_+-\omega_1)\big)e^{i\tau_c\beta_+}\big)\big], \end{aligned}$$

$$a_{27} = -\frac{16\tau_c}{m_1 n_2 - m_2 n_1}\left[K_c^4(m_1 - n_1) + c|d|^2\left(m_1\left(K_c - 1 + i(\beta_- - \omega_1)\right)e^{i\tau_c\beta_-} - n_1\left(K_c - 1 + i(\beta_+ - \omega_1)\right)e^{i\tau_c\beta_+}\right)\right].$$

Currently, we only give the normal forms in the cases above and hope that a detailed analysis of the dynamics can be conducted in future.

6. Conclusion

In our paper, we investigate the double Hopf bifurcation of the trivial equilibrium for delay-coupled limit cycle oscillators. Using the normal form method for functional differential equations (FDEs) in [28], and the center manifold theory in [36], we have obtained the universal unfolding of system (1.1) with two different pairs of pure imaginary eigenvalues and the complete dynamics near the double Hopf bifurcation.

The previous results have shown that such a delay coupling can cause amplitude death, especially death islands, whose margins are the critical lines of Hopf bifurcation. Our discussions indicate that the double Hopf bifurcation happens when two different Hopf bifurcation lines intersect. Thus, there exists more complex dynamics near the margins of the death islands except periodic solution, such as periodic motion and three-dimensional torus. But as the limitations of our knowledge, we only get the normal forms for strong resonant cases. It is our next research to analyze the fancy dynamics of strong resonance utilizing the normal forms.

References

[1] C. Wu, Synchronization in Coupled Chaotic Circuits and Systems, World Scientific Publishing Company, 2002.
[2] J. Gunawardena, Chemical Reaction Network Theory for In-Silico Biologists, Harvard University, MA, 2003.
[3] V. Gazi, K. Passino, Stability analysis of swarms, IEEE Trans. Automat. Control 48 (2003) 692–697.
[4] A. Jadbabaie, J. Lin, A. Morse, Coordination of groups of mobile autonomous agents using nearest neighbor rules, IEEE Trans. Automat. Control 48 (2003) 988–1001.
[5] R. Olfati-Saber, R. Murray, Consensus problems in networks of agents with switching topology and time-delays, IEEE Trans. Automat. Control 49 (2004) 1520–1533.
[6] L.O. Chua, T. Roska, Cellular Neural Networks and Visual Computing: Foundations and Applications, Cambridge University Press, Cambridge, 2002.
[7] J. Lu, Z. Ma, L. Li, Double delayed feedback control for the stabilization of unstable steady states in chaotic systems, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 3037–3045.
[8] A. Luo, A theory for synchronization of dynamical systems, Commun. Nonlinear Sci. Numer. Simul. 14 (2009) 1901–1951.
[9] F. Cucker, S. Smale, Emergent behavior in flocks, IEEE Trans. Automat. Control 52 (2007) 852–862.
[10] J.W.S. Rayleigh, The Theory of Sound, second ed., Dover, New York, 1945 (originally published in 1877–1878).
[11] D.G. Aronson, G.B. Ermentrout, N. Kopell, Amplitude response of coupled oscillators, Phys. D 41 (1990) 403–449.
[12] D.V.R. Reddy, A. Sen, G.L. Johnston, Time delay induced death in coupled limit cycle oscillators, Phys. Rev. Lett. 80 (1998) 5109–5112.
[13] F.M. Atay, Total and partial amplitude death in networks of diffusively coupled oscillators, Phys. D 183 (2003) 1–18.
[14] F.M. Atay, Complex Time-Delay Systems, Springer-Verlag, Berlin, 2010.
[15] F.M. Atay, Oscillator death in coupled functional differential equations near Hopf bifurcation, J. Differential Equations 221 (2006) 190–209.
[16] D.V.R. Reddy, A. Sen, G.L. Johnston, Time delay effects on coupled limit cycle oscillators at Hopf bifurcation, Phys. D 129 (1999) 15–34.
[17] Y. Song, J. Wei, Y. Yuan, Stability switches and Hopf bifurcations in a pair of delay-coupled oscillators, Nonlinear Sci. 17 (2007) 145–166.
[18] Y. Li, H. Wang, W. Jiang, Stability and Hopf bifurcation analysis in coupled limit cycle oscillators with time delay, Int. J. Innov. Comput. I 6 (2010) 1823–1832.
[19] W. Jiang, J. Wei, Bifurcation analysis in a limit cycle oscillator with delayed feedback, Chaos Solitons Fractals 23 (2005) 817–831.
[20] B. Niu, J. Wei, Stability and bifurcation analysis in an amplitude equation with delayed feedback, Chaos Solitons Fractals 37 (2008) 1362–1371.
[21] J. Xu, L. Pei, The nonresonant double Hopf bifurcation in delayed neural network, Int. J. Comput. Math. 85 (2008) 925–935.
[22] E. Knobloch, Normal-form coefficients for the nonresonant double Hopf-bifurcation, Phys. Lett. A 116 (1986) 365–369.
[23] W. Wang, Q. Zhang, Computation of the simplest normal form of a resonant double Hopf bifurcation system with the complex normal form method, Nonlinear Dynam. 57 (2009) 219–229.
[24] J. Xu, K.W. Chung, Double Hopf bifurcation with strong resonances in delayed systems with nonlinearities, Math. Probl. Eng. 2009 (2009) 1–17.
[25] Q. Bi, P. Yu, Double Hopf bifurcation and chaos of a nonlinear vibration system, Nonlinear Dynam. 19 (1999) 313–332.
[26] G. Revel, D.M. Alonso, J.L. Moiola, Interactions between oscillatory modes near a 2 : 3 resonant Hopf–Hopf bifurcation, Chaos 20 (2010) 043106.
[27] J. Xu, K.W. Chung, C.L. Chan, An efficient method for studying weak resonant double Hopf bifurcation in nonlinear systems with delayed feedbacks, SIAM J. Appl. Dyn. Syst. 6 (2007) 29–60.
[28] T. Faria, L.T. Magalhães, Normal forms for retarded functional differential equations and applications to Bogdanov–Takens singularity, J. Differential Equations 122 (1995) 201–224.
[29] S. Ma, Q. Lu, Z. Feng, Double Hopf bifurcation for van der Pol–Duffing oscillator with parametric delay feedback control, J. Math. Anal. Appl. 338 (2008) 993–1007.
[30] W. Jiang, Y. Yuan, Bogdanov–Takens singularity in van der Pol's oscillator with delayed feedback, Phys. D 227 (2007) 149–161.
[31] Y.A. Kuznetsov, Elements of Applied Bifurcation Theory, second ed., Springer-Verlag, New York, 1998.
[32] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.
[33] S. Guo, Y. Chen, J. Wu, Two-parameter bifurcations in a network of two neurons with multiple delays, J. Differential Equations 244 (2008) 444–486.
[34] H. Wang, W. Jiang, Hopf-pitchfork bifurcation in van der Pols oscillator with nonlinear delayed feedback, J. Math. Anal. Appl. 368 (2010) 9–18.
[35] W. Jiang, H. Wang, Hopf-transcritical bifurcation in retarded functional differential equations, Nonlinear Anal. 73 (2010) 3626–3640.
[36] J. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.